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# Study of Two Matrix Fractional Integrals by Using Differentiation under Fractional Integral Sign

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*Abstract:* **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional integral, we find the exact solutions of two matrix fractional integrals. Differentiation under fractional integral sign and a new multiplication of fractional analytic functions play important roles in this article. In fact, our results are generalizations of classical calculus results.**

*Keywords:* **Jumarie type of R-L fractional integral, matrix fractional integrals, differentiation under fractional integral sign, new multiplication, fractional analytic functions.**

## I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as physics, biology, mechanics, electrical engineering, viscoelasticity, control theory, modelling, economics, etc [1-15]. However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivative. Other useful definitions include Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie type of R-L fractional derivative to avoid non-zero fractional derivative of constant function [16-20].

In this paper, based on Jumarie's modified R-L fractional calculus, we obtain the exact solutions of the following two matrix $\alpha$-fractional integrals:

$$\left({}_0I_x^{\alpha}\right)\left[\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha m}\otimes_\alpha E_\alpha(pAx^\alpha)\otimes_\alpha cos_\alpha(qAx^\alpha)\right],$$

and

$$\left({}_0I_x^{\alpha}\right)\left[\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha m}\otimes_\alpha E_\alpha(pAx^\alpha)\otimes_\alpha sin_\alpha(qAx^\alpha)\right],$$

where $0<\alpha\leq 1$, $p,q$ are real numbers, $p^2+q^2\neq 0$, $m$ is a positive integer, and $A$ is a real matrix. Differentiation under fractional integral sign and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of classical calculus results.

## II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper and its properties.

**Definition 2.1** ([21])**:** Let $0<\alpha\leq 1$, and $x_0$ be a real number. The Jumarie type of Riemann-Liouville (R-L) $\alpha$-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \quad (1)$$

And the Jumarie type of Riemann-Liouville $\alpha$-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

**Proposition 2.2** ([22])**:** *If* $\alpha, \beta, x_0, C$ *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)[(x-x_0)^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

*and*

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

**Definition 2.3** ([23])**:** If $x, x_0$, and $a_k$ are real numbers for all $k$, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an $\alpha$-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing $x_0$, then we say that $f_\alpha(x^\alpha)$ is $\alpha$-fractional analytic at $x_0$. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is $\alpha$-fractional analytic at every point in open interval $(a,b)$, then $f_\alpha$ is called an $\alpha$-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

**Definition 2.4** ([24])**:** Let $0 < \alpha \leq 1$, and $x_0$ be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two $\alpha$-fractional analytic functions defined on an interval containing $x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

**Definition 2.5** ([25])**:** Let $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two $\alpha$-fractional analytic functions. Then $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha n} = f_\alpha(x^\alpha)\otimes_\alpha \cdots \otimes_\alpha f_\alpha(x^\alpha)$ is called the $n$th power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) = 1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha -1}$.

**Definition 2.6** ([26])**:** If $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two $\alpha$-fractional analytic functions defined on an interval containing $x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} = \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} = \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\big(g_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\big(f_\alpha(x^\alpha)\big) = \sum_{n=0}^{\infty} \frac{b_n}{n!} \big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

**Definition 2.7** ([27])**:** If $0 < \alpha \le 1$, $x$ is a real number, and $A$ is a real matrix. Then the matrix $\alpha$-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes n}. \tag{13}$$

And the matrix $\alpha$-fractional cosine and matrix $\alpha$-fractional sine function are defined as follows:

$$cos_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^{2n} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2n}, \tag{14}$$

and

$$sin_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^{2n+1} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{15}$$

**Theorem 2.8** (matrix fractional Euler's formula) ([28])**:** *If $0 < \alpha \le 1$, $i = \sqrt{-1}$, and $A$ is a real matrix, then*

$$E_\alpha(iAx^\alpha) = cos_\alpha(Ax^\alpha) + isin_\alpha(Ax^\alpha). \tag{16}$$

**Notation 2.9:** If $s$ is a real number, the largest integer less than or equal to $s$ is denoted by $\lfloor s \rfloor$.

**Notation 2.10:** If the complex number $z = p + iq$, where $p, q$ are real numbers, $p$ the real part of $z$, is denoted by $\mathrm{Re}(z)$; $q$ the imaginary part of $z$, is denoted by $\mathrm{Im}(z)$.

**Notation 2.11:** If $r, s$ are positive integers, $r \le s$, then $\binom{s}{r} = \frac{s!}{r!(s-r)!}$. In addition, we define $\binom{s}{0} = 1$, and $\binom{0}{r} = 0$ for all positive integers $r$.

**Theorem 2.12** (differentiation under fractional integral sign) ([29])**:** *If $0 < \alpha \le 1$, $t$ is a nonzero real variable, and $f_\alpha(x^\alpha)$ is a $\alpha$-fractional analytic function at $x = 0$, then*

$$\frac{d}{dt}\left({}_0I_x^\alpha\right)[f_\alpha(t, x^\alpha)] = \left({}_0I_x^\alpha\right)\left[\frac{d}{dt} f_\alpha(t, x^\alpha)\right].$$

(17)

## III. MAIN RESULTS

In this section, we find the exact solutions of two matrix fractional integrals by using differentiation under fractional integral sign. At first, three lemmas are needed.

**Lemma 3.1:** *If $p, q$ are real numbers, and $k$ is a positive integer, then*

$$(p - iq)^k + (p + iq)^k = 2 \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} (-1)^j p^{k-2j} q^{2j}, \tag{18}$$

$$(p - iq)^k - (p + iq)^k = -2i \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} (-1)^l p^{k-2l-1} q^{2l+1}, \tag{19}$$

**Proof** $\quad (p - iq)^k + (p + iq)^k$

$$= \sum_{m=0}^{k} \binom{k}{m} p^{k-m} (-iq)^m + \sum_{m=0}^{k} \binom{k}{m} p^{k-m} (iq)^m$$

$$= \sum_{m=0}^{k} \binom{k}{m} p^{k-m} q^m [(-1)^m + 1] i^m$$

$$= 2\sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} (-1)^j p^{k-2j} q^{2j} .$$

Similarly,

$$(p-iq)^k - (p+iq)^k$$

$$= \sum_{m=0}^{k} \binom{k}{m} p^{k-m} (-iq)^m - \sum_{m=0}^{k} \binom{k}{m} p^{k-m} (iq)^m$$

$$= \sum_{m=0}^{k} \binom{k}{m} p^{k-m} q^m [(-1)^m - 1] i^m$$

$$= -2i \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} (-1)^l p^{k-2l-1} q^{2l+1} .$$ q.e.d.

**Lemma 3.2:** *Let* $0 < \alpha \leq 1$, $p, q$ *be real numbers, and* $k$ *be a positive integer, then*

$$\frac{d^k}{dp^k}\left(\frac{p}{p^2+q^2}\right) = \frac{(-1)^k k!\cdot}{(p^2+q^2)^{k+1}} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} (-1)^j p^{k-2j} q^{2j}, \tag{20}$$

$$\frac{d^k}{dp^k}\left(\frac{q}{p^2+q^2}\right) = \frac{(-1)^k k!\cdot}{(p^2+q^2)^{k+1}} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} (-1)^l p^{k-2l-1} q^{2l+1}, \tag{21}$$

**Proof** $$\frac{d^k}{dp^k}\left(\frac{p}{p^2+q^2}\right)$$

$$= \frac{d^k}{dp^k}\left(\frac{1/2}{p+iq} + \frac{1/2}{p-iq}\right)$$

$$= \frac{1}{2}(-1)^k k! \left[\frac{1}{(p+iq)^{k+1}} + \frac{1}{(p-iq)^{k+1}}\right]$$

$$= \frac{1}{2}(-1)^k k! \left[\frac{(p-iq)^{k+1} + (p+iq)^{k+1}}{(p^2+q^2)^{k+1}}\right]$$

$$= \frac{(-1)^k k!\cdot}{(p^2+q^2)^{k+1}} \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} (-1)^j p^{k-2j} q^{2j} .$$

And

$$\frac{d^k}{dp^k}\left(\frac{q}{p^2+q^2}\right)$$

$$= \frac{d^k}{dp^k}\left(\frac{-1/2i}{p+iq} + \frac{1/2i}{p-iq}\right)$$

$$= -\frac{1}{2i}\frac{d^k}{dp^k}\left(\frac{1}{p+iq} - \frac{1}{p-iq}\right)$$

$$= -\frac{1}{2i}(-1)^k k! \left[\frac{1}{(p+iq)^{k+1}} - \frac{1}{(p-iq)^{k+1}}\right]$$

$$= -\frac{1}{2i}(-1)^k k! \left[\frac{(p-iq)^{k+1} - (p+iq)^{k+1}}{(p^2+q^2)^{k+1}}\right]$$

$$= \frac{(-1)^k k!\cdot}{(p^2+q^2)^{k+1}} \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} (-1)^l p^{k-2l-1} q^{2l+1} .$$ q.e.d.

**Lemma 3.3:** *Let* $0 < \alpha \leq 1$, $p, q$ *be real numbers,* $p^2 + q^2 \neq 0$, *and* $A$ *be a real matrix, then the matrix* $\alpha$*-fractional integrals*

$$\left({}_0I_x^{\alpha}\right)\left[E_\alpha(pAx^\alpha)\otimes_\alpha cos_\alpha(qAx^\alpha)\right] = E_\alpha(pAx^\alpha)\otimes_\alpha \left[\frac{p}{p^2+q^2} cos_\alpha(qAx^\alpha) + \frac{q}{p^2+q^2} sin_\alpha(qAx^\alpha)\right], \tag{22}$$

$$\left({}_0I_x^{\alpha}\right)\left[E_\alpha(pAx^\alpha)\otimes_\alpha sin_\alpha(qAx^\alpha)\right] = E_\alpha(pAx^\alpha)\otimes_\alpha \left[\frac{p}{p^2+q^2} sin_\alpha(qAx^\alpha) - \frac{q}{p^2+q^2} cos_\alpha(qAx^\alpha)\right]. \tag{23}$$

**Proof** $\left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} cos_{\alpha}(qAx^{\alpha})]$

$$= \left({}_{0}I_{x}^{\alpha}\right)\left[\mathrm{Re}[E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} E_{\alpha}(iqAx^{\alpha})]\right]$$

$$= \mathrm{Re}\{\left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}((p+iq)Ax^{\alpha})]\}$$

$$= \mathrm{Re}\left\{\frac{1}{p+iq}E_{\alpha}((p+iq)Ax^{\alpha})\right\}$$

$$= \mathrm{Re}\left\{\frac{p-iq}{p^2+q^2}E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} E_{\alpha}(iqAx^{\alpha})\right\}$$

$$= \mathrm{Re}\left\{\frac{p-iq}{p^2+q^2}E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}[cos_{\alpha}(qAx^{\alpha})+isin_{\alpha}(qAx^{\alpha})]\right\}$$

$$= E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}\left[\frac{p}{p^2+q^2}cos_{\alpha}(qAx^{\alpha})+\frac{q}{p^2+q^2}sin_{\alpha}(qAx^{\alpha})\right].$$

And

$$\left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} sin_{\alpha}(qAx^{\alpha})]$$

$$= \left({}_{0}I_{x}^{\alpha}\right)\left[\mathrm{Im}[E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} E_{\alpha}(iqAx^{\alpha})]\right]$$

$$= \mathrm{Im}\{\left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}((p+iq)Ax^{\alpha})]\}$$

$$= \mathrm{Im}\left\{\frac{1}{p+iq}E_{\alpha}((p+iq)Ax^{\alpha})\right\}$$

$$= \mathrm{Im}\left\{\frac{p-iq}{p^2+q^2}E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} E_{\alpha}(iqAx^{\alpha})\right\}$$

$$= \mathrm{Im}\left\{\frac{p-iq}{p^2+q^2}E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}[cos_{\alpha}(qAx^{\alpha})+isin_{\alpha}(qAx^{\alpha})]\right\}$$

$$= E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}\left[\frac{p}{p^2+q^2}sin_{\alpha}(qAx^{\alpha})-\frac{q}{p^2+q^2}cos_{\alpha}(qAx^{\alpha})\right].$$ q.e.d.

**Theorem 3.4:** *If $0<\alpha\le 1$, $p,q$ are real numbers, $p^2+q^2\neq 0$, $m$ is a positive integer, and $A$ is a real matrix, then the matrix α-fractional integrals*

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} m}\otimes_{\alpha} E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} cos_{\alpha}(qAx^{\alpha})\right]$$

$$= \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k k!}{(p^2+q^2)^{k+1}}\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-k)}\otimes_{\alpha} E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}\left[\sum_{j=0}^{[k/2]}\binom{k}{2j}(-1)^j p^{k-2j}q^{2j}\cdot cos_{\alpha}(qAx^{\alpha})+\right.$$
*l=0(k−1)/2k2l+1(−1)lpk−2l−1q2l+1·sinαqAxα* , (24)

And

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} m}\otimes_{\alpha} E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} sin_{\alpha}(qAx^{\alpha})\right]$$

$$= \sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k k!}{(p^2+q^2)^{k+1}}\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(m-k)}\otimes_{\alpha} E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}\left[\sum_{j=0}^{[k/2]}\binom{k}{2j}(-1)^j p^{k-2j}q^{2j}\cdot sin_{\alpha}(qAx^{\alpha})-\right.$$
*l=0(k−1)/2k2l+1(−1)lpk−2l−1q2l+1·cosαqAxα* . (25)

**Proof** $\left({}_{0}I_{x}^{\alpha}\right)\left[\left(A\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} m}\otimes_{\alpha} E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} cos_{\alpha}(qAx^{\alpha})\right]$

$$= \frac{d^m}{dp^m}\left[\left({}_{0}I_{x}^{\alpha}\right)[E_{\alpha}(pAx^{\alpha})\otimes_{\alpha} cos_{\alpha}(qAx^{\alpha})]\right]$$ (by differentiation under fractional integral sign)

$$= \frac{d^m}{dp^m}\left[E_{\alpha}(pAx^{\alpha})\otimes_{\alpha}\left[\frac{p}{p^2+q^2}cos_{\alpha}(qAx^{\alpha})+\frac{q}{p^2+q^2}sin_{\alpha}(qAx^{\alpha})\right]\right]$$ (by Lemma 3.3)

$$=\sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}}{dp^{m-k}}[E_\alpha(pAx^\alpha)]\otimes_\alpha \frac{d^k}{dp^k}\left[\frac{p}{p^2+q^2}cos_\alpha(qAx^\alpha)+\frac{q}{p^2+q^2}sin_\alpha(qAx^\alpha)\right]$$

$$=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k k!}{(p^2+q^2)^{k+1}}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-k)}\otimes_\alpha E_\alpha(pAx^\alpha)\otimes_\alpha\left[\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}(-1)^j p^{k-2j}q^{2j}\cdot cos_\alpha(qAx^\alpha)+\right.$$

*l=0(k−1)/2k2l+1(−1)lpk−2l−1q2l+1·sinαqAxα*. (by Lemma 3.2)

On the other hand,

$$\left({}_0I_x^\alpha\right)\left[\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha m}\otimes_\alpha E_\alpha(pAx^\alpha)\otimes_\alpha sin_\alpha(qAx^\alpha)\right]$$

$$=\frac{d^m}{dp^m}\left[\left({}_0I_x^\alpha\right)\left[E_\alpha(pAx^\alpha)\otimes_\alpha sin_\alpha(qAx^\alpha)\right]\right]$$ (by differentiation under fractional integral sign)

$$=\frac{d^m}{dp^m}\left[E_\alpha(pAx^\alpha)\otimes_\alpha\left[\frac{p}{p^2+q^2}sin_\alpha(qAx^\alpha)-\frac{q}{p^2+q^2}cos_\alpha(qAx^\alpha)\right]\right]$$ (by Lemma 3.3)

$$=\sum_{k=0}^{m}\binom{m}{k}\frac{d^{m-k}}{dp^{m-k}}[E_\alpha(pAx^\alpha)]\otimes_\alpha \frac{d^k}{dp^k}\left[\frac{p}{p^2+q^2}sin_\alpha(qAx^\alpha)-\frac{q}{p^2+q^2}cos_\alpha(qAx^\alpha)\right]$$

$$=\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k k!}{(p^2+q^2)^{k+1}}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(m-k)}\otimes_\alpha E_\alpha(pAx^\alpha)\otimes_\alpha\left[\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}(-1)^j p^{k-2j}q^{2j}\cdot sin_\alpha(qAx^\alpha)-\right.$$

*l=0(k−1)/2k2l+1(−1)lpk−2l−1q2l+1·cosαqAxα*. (by Lemma 3.2) q.e.d.

## IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we find the exact solutions of two fractional integrals. Integration by parts for fractional calculus and a new multiplication of fractional analytic functions play important roles in this article. In fact, the major results we obtained are natural generalizations of the results in classical calculus. In the future, we will continue to use our methods to study the problems in applied mathematics and fractional differential equations.

## REFERENCES

[1] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[2] R. L. Magin, Fractional calculus models of complex dynamics in biological tissues, Computers & Mathematics with Applications, vol. 59, no. 5, pp. 1586-1593, 2010.

[3] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[4] F. C. Meral, T. J. Royston, R. Magin, Fractional calculus in viscoelasticity: an experimental study, Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 4, pp. 939-945, 2010.

[5] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[6] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol. 2, Application. Springer, 2013.

[7] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, Vol. 8, No. 5, 660, 2020.

[8] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[9] A. Carpinteri, F. Mainardi, (Eds.), Fractals and fractional calculus in continuum mechanics, Springer, Wien, 1997.

[10] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, vol. 14, no. 9, pp. 1587-1596, 2008.

[11] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[12] T. Sandev, R. Metzler, & Ž. Tomovski, Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative, Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 25, 255203, 2011.

[13] J. P. Yan, C. P. Li, On chaos synchronization of fractional differential equations, Chaos, Solitons & Fractals, vol. 32, pp. 725-735, 2007.

[14] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[15] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[16] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[17] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[18] I. Podlubny, Fractional Differential Equations; Academic Press: New York, NY, USA, 1999.

[19] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[20] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.[21] C. -H. Yu, Study on some properties of fractional analytic function, International Journal of Mechanical and Industrial Technology, vol. 10, no. 1, pp. 31-35, 2022.

[21] C. -H. Yu, A Study of some fractional differential problems, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 1, pp. 52-58, 2022.

[22] C. -H. Yu, Exact solution of linear system of fractional differential equations with constant coefficients, International Journal of Mechanical and Industrial Technology, vol. 10, no. 2, pp. 1-7, 2022.

[23] C. -H. Yu, Application of differentiation under fractional integral sign, International Journal of Mathematics and Physical Sciences Research, vol. 10, no. 2, pp. 40-46, 2022.

[24] C. -H. Yu, Fractional differential problem of some fractional trigonometric functions, International Journal of Interdisciplinary Research and Innovations, vol. 10, no. 4, pp. 48-53, 2022.

[25] C. -H. Yu, Infinite series expressions for the values of some fractional analytic functions, International Journal of Interdisciplinary Research and Innovations, vol. 11, no. 1, pp. 80-85, 2023.

[26] C. -H. Yu, Study of two fractional integrals, International Journal of Novel Research in Physics Chemistry & Mathematics, vol. 10, no. 2, pp. 1-6, 2023.

[27] C. -H. Yu, Application of integration by parts for fractional calculus in solving two types of fractional definite integrals, International Journal of Novel Research in Electrical and Mechanical Engineering, vol. 10, no. 1, pp. 79-84, 2023.

[28] C. -H, Yu, Evaluating fractional derivatives of two matrix fractional functions based on Jumarie type of Riemann-Liouville fractional derivative, International Journal of Engineering Research and Reviews, vol. 12, no. 4, pp. 39-43, 2024.

[29] C. -H, Yu, Studying three types of matrix fractional integrals, International Journal of Interdisciplinary Research and Innovations, vol. 12, no. 4, pp. 35-39, 2024.